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New Approach for System Reliability-Based Design Optimization

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An efficient approach for reliability-based design optimization (RBDO) of series systems is presented. A modified formulation of the RBDO problem is employed in which the reliabilities of the failure modes of a system are included in the set of the design variables. This allows for an optimal apportionment of the reliability of a system among its failure modes. A sequential optimization and reliability assessment method is used to efficiently determine the optimum design. Here, the constraints on the reliabilities of the failure modes of the RBDO problem are replaced with approximate deterministic constraints. The proposed approach is demonstrated on two example problems that have been solved in previous studies without optimizing the required reliability levels of the failure modes. The first example performs RBDO to a cantilever beam with a rectangular cross section under lateral and vertical loads. The constraints are on the strength and the maximum allowable displacement. The second example performs RBDO to a cable-stayed box girder with five failure modes. Compared to the designs found by previous studies, the new approach finds designs with lower mass but without reducing the system reliability.

Introduction

CONTINUOUS efforts of the aerospace, automotive, and shipbuilding industries to minimize the structural weight of vehicles while maintaining an acceptable level of safety have made it important to develop reliable and efficient design optimization techniques. In many structural design situations the operating conditions, material properties, and the geometry are uncertain. In this paper, the term "uncertainty" refers to both random uncertainty (variability) and epistemic or reducible uncertainty (uncertainty because of a lack of knowledge). Optimizing the design in the face of uncertainty requires special consideration to quantify and control the risk of failure. For this purpose, these industries are trying to incorporate risk-based optimization techniques in their design systems.

Reliability-based design optimization (RBDO) is being increasingly accepted by the industry. RBDO uses probability theory and statistics to model uncertainty and to determine the probability of failure and performs optimization to find the design with minimum weight, which satisfies the limits of the allowable probability of failure. A designer faces many challenges when applying RBDO to many real-life problems. Finding the probability of failure of a given design requires repeating the structural analysis for different sets of the values of the random variables, which can be computationally expensive, especially when using finite element analysis.[1–3] Also, the computational expense might be grossly compounded when calculating the probability of failure of different designs during the search for an optimum reliability-based design. In addition, the performance function (a function that is positive if a structure survives and negative if it fails) can be highly nonlinear and can have derivative discontinuity that can cause the optimization process to fail.[4–7] This is because the reliability of the design (which is a function of the performance function) can be dramatically modified even for a small change in some design variables, and for a design with multiple failure modes this can produce numerical discontinuity in the system reliability.

Two methods are commonly used to calculate the reliability of a design: Monte Carlo simulation techniques[1–3,8–11] and analytical reliability approximate methods [e.g., the first-order reliability method (FORM), the second-order reliability method, and the advanced mean-value method]. The analytical reliability approximation methods find the most probable point (MPP) for the design, which is an optimization problem.[1–3,12–15] Accordingly, to perform RBDO two nested optimization loops can be employed: an inner loop to calculate the reliability and an outer loop to perform design optimization. In the past two decades, researchers have used this nested optimization loops technique to perform RBDO, which can be very expensive and can lead to divergence.[4–7] As a result, attempts were made to reduce the number of required structural analyses during RBDO and to overcome the problems of high nonlinearity and derivative discontinuity of the reliability function. For example, some researchers have developed efficient surrogate models (e.g., response surface polynomials and neural networks) to approximate the structural response and then used these approximate models to perform RBDO.[16–21] Others approximated the reliability function itself.[22,23] Yet, these approximate models can be computationally very expensive to develop or can be highly inaccurate,[24] especially when there are many random variables (e.g., more than 10).

Some researchers have tried to combine the optimization process and the reliability search into one optimization loop. This goal requires overcoming many challenges. Madsen and Hansen[25] have found that a multiobjective function RBDO formulation might need 50% more iterations than a nested RBDO formulation for the same problem. Also, Royset et al.[5] proposed reformulating the RBDO to an outer deterministic semi-infinite optimization loop and a separate inner loop to calculate the reliability of the design by any reliability method (for semi-infinite optimization algorithms, see Polak[26]). The efficiency and robustness of the semi-infinite algorithm used by Royset et al. was not compared to other widely used optimization methods. Also, Royset et al. have noted that the success of their proposed method depends on how the reliability calculations modify the optimization subproblem. Kharamanda et al.[27] proposed incorporating the reliability calculation optimization loop into the

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design optimization loop and performed optimization in a hybrid optimization space with design variables those of the original design optimization problem plus the random variables of the reliability assessment problem. The new objective function is the product of the design cost function (cost or weight) times the objective function of the optimization loop for finding the MPP. Kharamanda et al.[27] reported a fivefold reduction in the number of evaluations compared to a nested RBDO. In our opinion, Kharamanda et al.[27] did not show that the new single-loop optimization problem has the same solution as the original, two-loop optimization problem.

Finally, some researchers have proposed unilevel methods[28,29] to reduce the computational cost of RBDO by replacing the inner optimization loop for reliability assessment by the first-order Karush–Kuhn–Tucker (KKT) optimality conditions for the MPPs of the reliability constraints. The KKT condition for a reliability constraint dictates that the gradient of the performance function and the position vector at the MPP be collinear in the space of the reduced variables. These researchers used the conditions that must be satisfied by the reliability analysis as equality constraints that must be satisfied by the RBDO problem. Thus the optimization is performed in a single loop that contains approximate reliability constraints along with the KKT conditions as equality constraints instead of the actual reliability constraints. The method of Agarwal et al.[28] increases the dimension of the optimization problem as each random variable is varied differently in each approximate reliability constraint in addition to the KKT equality constraints. Hence, the computational savings of Agarwal's method can be just around 50% of the computational cost of the nested optimization loops. The unilevel method of Liang et al.[29] does not increase the number of design variables of the optimization problem.

Alternatively, attempts were made to exploit the fact that deterministic optimization requires much less computation than RBDO. Although there is rarely an explicit relation between the safety factors and the reliability of a system,[30] researchers have found that they might be able to derive approximate probabilistic safety factors from the reliability calculations and use these factors to replace the probabilistic constraints with equivalent deterministic constraints.[24,31–34] This decouples the reliability calculations from the design optimization and allows solving the RBDO problem by solving a sequence of deterministic optimization problems. This is the main idea behind the sequential optimization with reliability-based factors of safety (SORFS) RBDO methods. In this way, the optimum design search can be performed deterministically, instead of using the actual reliability constraints and searching in the probability space, thereby reducing substantially the number of performance function evaluations.

As just mentioned, SORFS methods[31–34] replace the constraints on the failure probabilities of the modes of a system with equivalent deterministic constraints. These constraints dictate that the performance functions of the modes be nonnegative (that is, the system should not fail under these modes) at some checking point representing a set of critical values of the random variables. The checking point is an approximation of the MPP for the minimum allowable value of the safety index. Figure 1 shows an example of an approximation of the MPP for a failure mode as a function of the safety index. SORFS methods require iterations (Fig. 2); each iteration consists of 1) reliability analyses to find the MPPs for the critical failure modes, 2) approximations of the MPPs as functions of the safety indices, and 3) solution of the approximate deterministic optimization problem to find the optimum values of the design variables and the minimum allowable values of the safety indices. An inverse MPP search algorithm, which is also called performance measure analysis (PMA),[33–38] is often employed to find the MPP for a particular value of the safety index.

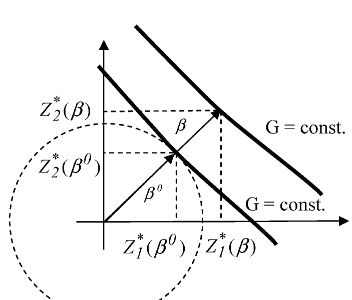

Fig. 1 Approximation of the coordinates of the MPP as function of the safety index.

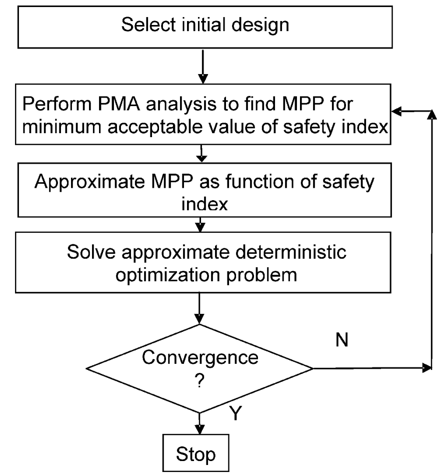

Fig. 2 Algorithm of a SORFS method.

The preceding methods can handle RBDO problems in which each critical failure mode of a system has a predetermined minimum acceptable safety level. Thus the user of the method has to decide the minimum safety level of each failure mode instead of letting the optimizer determine these levels to achieve a predetermined minimum acceptable system reliability. We believe that we can obtain considerably better designs by allowing the optimizer to determine the required safety levels of the failure modes, instead of asking the user to select them. Hence, in the following we will present a modification of SORFS methods for systems that optimizes the safety levels of the failure modes.

The rest of the paper presents a formulation of the RBDO problem where the maximum allowable failure probabilities of the failure modes are included in the set of design variables. The algorithm for the new approach is also presented. The approach is demonstrated on two example problems that have been solved using conventional RBDO approaches in previous studies by other researchers. The first example relates to RBDO of a cantilever beam with rectangular cross section under random vertical and horizontal tip loads. The constraints are on the strength and the maximum displacement. The second example relates to the RBDO of a cable-stayed box girder with five constraints on the strengths of the beam and the cable and on the tip displacement. The new approach finds designs with lower mass than the designs found in the previous studies but without reducing their system reliability.

Modified Sequential Approach for RBDO of Systems

Formulation

The general RBDO problem formulation for a series system is as follows:

Find the design with the least cost (or weight) such that the system probability of failure is less than or equal to a maximum allowable value:

$$\text{Find} \quad \boldsymbol{d}$$

$$\text{to min:} \quad F(\boldsymbol{d})$$

$$\text{s.t.:} \quad P_{\text{sys}} = P\left[\bigcup_{i=1}^{m} G_i(\boldsymbol{d}, \boldsymbol{X}) \le 0\right] \le P_f^{\text{all}} \tag{1}$$

where $\boldsymbol{d}$ is the vector of design variables, $F(\,)$ is the objective function (e.g., the weight or cost),

$$P_{\text{sys}} = P\left[\bigcup_{i=1}^{m} G_i(\boldsymbol{d}, \boldsymbol{X}) \le 0\right]$$

is the system probability of failure under the performance functions $G_i(\boldsymbol{d}, \boldsymbol{X}), i = 1, \ldots, m, \boldsymbol{X}$ is the vector of random variables, and P_f^{all} is the maximum allowable system probability of failure. Note that the design variables are deterministic, and they can include the distribution parameters of the random variables (e.g., the mean value of the thickness of a plate).

The failure probability of a series system is approximated by the sum of the failure probabilities of the failure modes (using Ditlevsen's first-order upper bound), which means that we are conservatively ignoring the intersection between failure modes. If we consider the maximum allowable probabilities of the failure modes $P_{f_i}^{\text{all}}$ as design variables, then the system RBDO formulation in Eq. (1) becomes

$$\begin{aligned} &\text{Find} \quad \boldsymbol{d}, P_{f_i}^{\text{all}} \\ &\text{to min:} \quad F(\boldsymbol{d}) \\ &\text{s.t.:} \quad P[G_i(\boldsymbol{d}, \boldsymbol{X}) \le 0] \le P_{f_i}^{\text{all}}, \qquad i = 1, \ldots, m \\ &P_{\text{sys}} = \sum_{i=1}^{m} P_{f_i}^{\text{all}} \le P_{f_i}^{\text{all}} \end{aligned} \tag{2}$$

In this formulation, the optimizer should determine the optimum values of the maximum allowable failure probabilities of the failure modes, besides the values of the original design variables $\boldsymbol{d}$. The constraints on the failure probabilities of the modes and the overall system failure probability can be written in terms of the safety indices of the failure modes β_i. Then the preceding optimization problem formulation becomes

$$\begin{aligned} &\text{Find} \quad \boldsymbol{d}, \beta_i^{\text{all}} \\ &\text{to min:} \quad F(\boldsymbol{d}) \\ &\text{s.t.:} \quad \beta_i \ge \beta_i^{\text{all}}, \qquad i = 1, \ldots, m \\ &P_{\text{sys}} = \sum_{i=1}^{m} \Phi\left(-\beta_i^{\text{all}}\right) \le P_f^{\text{all}} \end{aligned} \tag{3}$$

where Φ is the cumulative probability distribution of a standard normal random variable (zero mean, unit standard deviation) and $\Phi(-\beta_i^{\text{all}}) = P_{f_i}$. Symbol β_i^{all} denotes the minimum allowable value of the safety index of the ith failure mode. An equivalent performance measure approach formulation (PMA) to formulation (3) is as follows:

$$\begin{aligned} &\text{Find} \quad \boldsymbol{d}, \beta_i^{\text{all}} \\ &\text{to min:} \quad F(\boldsymbol{d}) \\ &\text{s.t.:} \quad G_i\left[\boldsymbol{d}, Z_i^*\left(\beta_i^{\text{all}}\right)\right] \ge 0, \qquad i = 1, \ldots, m \\ &\sum_{i=1}^{m} \Phi\left(-\beta_i^{\text{all}}\right) \le P_f^{\text{all}} \end{aligned} \tag{4}$$

$\boldsymbol{Z}_1^*, \ldots, \boldsymbol{Z}_m^*$ are the values of the reduced random variables at the most probable point for each failure mode, where $\boldsymbol{X}_i^* = \boldsymbol{T}(\boldsymbol{Z}_i^*), i = 1, \ldots, m$ and $\boldsymbol{T}$ is the transformation from the space of the reduced random variables to the space of the original random variables $\boldsymbol{X}_1^*, \ldots, \boldsymbol{X}_m^*$. $\boldsymbol{Z}_i^*$ is a vector with n components, where n is the number of random variables. In the PMA formulation, instead of checking if the minimum distance of the MPP from the origin is greater than the minimum allowable value of the safety index, we check if the performance function is nonnegative at the MPP, $\boldsymbol{Z}_i^*(\beta_i^{\text{all}})$. For each mode, the MPP is found for the minimum allowable value of the safety index. It is expensive to determine the MPPs of the failure modes repeatedly. Instead we approximate the coordinates of the most probable point $\boldsymbol{Z}_i^*(\beta_i)$ as a function of the value of the safety index β_i, given the MPP, $\boldsymbol{Z}_i^*(\beta_i^0)$, for a baseline value of the safety index β_i^0 as follows (Fig. 1):

$$\boldsymbol{Z}_i^*(\beta_i) = \left(\beta_i \big/ \beta_i^0\right)\boldsymbol{Z}_i^*\left(\beta_i^0\right) \tag{5}$$

This approximation allows recasting the system RDBO problem formulation as a deterministic optimization problem as follows:

$$\begin{aligned} &\text{Find} \quad \boldsymbol{d}, \beta_i^{\text{all}}, \qquad i = 1, \ldots, m \\ &\text{to min:} \quad F(\boldsymbol{d}) \\ &\text{s.t.:} \quad G_i\left[\boldsymbol{d}, \boldsymbol{Z}_i^*\left(\beta_i^0\right)\frac{\beta_i^{\text{all}}}{\beta_i^0}\right] \ge 0, \qquad i = 1, \ldots, m \\ &\sum_{i=1}^{m} \Phi\left(-\beta_i^{\text{all}}\right) \le P_f^{\text{all}} \end{aligned} \tag{6}$$

The solution of Eq. (6) is a design whose failure modes have safety indices approximately equal to β_i^{all}. These approximate values are called herein "projected values of the safety indices."

The approximation of the design point as a function of the safety index in Eq. (5) is only valid in a trust region around the baseline value of the safety index. The progress of the optimization should be monitored in each iteration, and the change in the value of the safety index should be constrained within some move limit to remain in the trust region. Available methods for optimization using trust regions can be used for this purpose.[39–41]

Algorithm

Figure 3 describes the system RBDO algorithm. Each iteration of this algorithm consists of three operations: 1) FORM reliability analysis of the initial design or of the design obtained from the previous iteration to check if this design has acceptable reliability, 2) PMA reliability analysis of the design to determine the MPPs of the failure modes $\boldsymbol{Z}_i^*(\beta_i^0)$, and 3) approximate deterministic optimization to update the optimum design and find the maximum allowable values of the safety indices of the failure modes β_i^{all}.

First, we perform deterministic optimization with some factor of safety to find an initial design. Then we perform FORM reliability analysis of the deterministic optimum. At this stage we can identify the noncritical failure modes, that is, those failure modes of the deterministic optimum design that do not affect significantly the system reliability because they have negligible probability of failure compared to the failure probabilities of the remaining critical modes. The minimum allowable values of the safety indices of these modes are removed from the set of design variables to facilitate convergence. Then we perform an inverse reliability analysis (PMA) of the deterministic optimum assuming equal probabilities of failure, which are determined as follows:

$$P_{f_i}^{\text{all}} = P_f^{\text{all}} \big/ m_c \tag{6a}$$

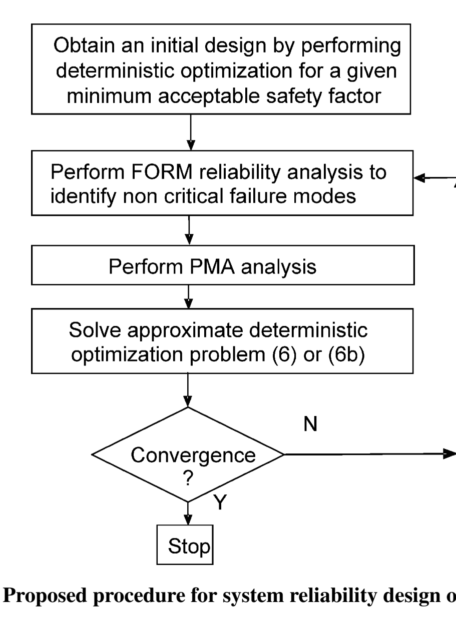

Fig. 3 Proposed procedure for system reliability design optimization.

where m_c is the number of critical failure modes. Finally, we perform approximate deterministic optimization considering the minimum allowable values of the safety indices of the m_c failure modes $\beta_1^{\text{all}}, \ldots, \beta_{m_c}^{\text{all}}$ as design variables in addition to the original design variables. Now the optimizer seeks both the optimum values of the design variables and the optimum target values of the safety indices to minimize the objective function (i.e., cost or weight), and the optimization problem is given by Eq. (6b). In this case, the optimization problem formulation (6) becomes

$$\begin{aligned} &\text{Find} \quad \boldsymbol{d}, \beta_i^{\text{all}}, \quad i = 1, \ldots, m_c \\ &\text{to min:} \quad F(\boldsymbol{d}) \\ &\text{s.t.:} \quad G_i\left[\boldsymbol{d}, \boldsymbol{Z}_i^*\left(\beta_i^0\right)\frac{\beta_i^{\text{all}}}{\beta_i^0}\right] \geq 0, \quad i = 1, \ldots, m_c \\ &\sum_{i=1}^{m_c} \Phi\left(-\beta_i^{\text{all}}\right) \leq P_f^{\text{all}} \end{aligned} \tag{6b}$$

The differences between formulations (6) and (6b) are in the number of design variables in the first equation, in the number of failure modes for which the value of the performance function is checked in the third equation, and in the number of failure modes considered when checking the system probability of failure in the last equation.

Once we have found the optimum, we check the failure probabilities of all failure modes (including the noncritical ones) using FORM. At this step, we can remove (or add) the minimum allowable values of the safety indices of some failure modes with low (high) failure probabilities from the set of design variables. Then we perform PMA analysis for the optimum values of the minimum allowable values of the safety indices found from the deterministic optimization. Finally, we solve the approximate deterministic optimization problem again. We repeat this process until convergence.

Example 1: Design of a Cantilever Beam with Rectangular Cross Section

An example presented by Wu et al.[32] is reconsidered here to determine whether the proposed RBDO method can find a better optimum design (less weight for the same system reliability or higher system reliability for the same weight) than the one found by Wu et al., in which the probabilities of the failure modes were not optimized. Also, we aim to examine the convergence of the method and its handling of the nonlinear constraints.

The cantilever beam shown in Fig. 4 has two failure modes, yielding and excessive deflection. The objective is to minimize the weight of the beam by controlling the cross-section dimensions (the width w and the thickness t).

The applied lateral loads X and Y, the yield strength R, and the modulus of elasticity E are random. All of these quantities are assumed normally distributed and statistically independent, with the mean values and coefficients of variation (COV) given in Table 1.

Table 1 Mean values and COVs for the random quantities

Quantity	Mean value	COV, %
Lateral horizontal load X, lb (N)	500 (889.6)	20
Lateral vertical load Y, lb (N)	1,000 (4,448)	10
Yield strength R, psi (N/m^2)	40,000 (2.76E8)	5
Modulus of elasticity E, psi (N/m^2)	2.9E7 (2E11)	5

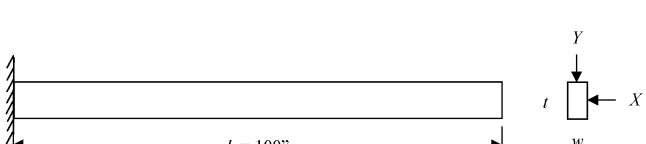

Fig. 4 Cantilever beam geometry and loads.

COV of a random variable is the standard deviation normalized by the mean value.

The two failure modes are yielding of the fixed-end corner of the beam [Eq. (7)] and excessive displacement of the tip [Eq. (8)]:

$$G_1(R, X, Y, w, t) = R - \left(\frac{600}{wt^2}Y + \frac{600}{w^2t}\right)X \tag{7}$$

$$G_1(E, X, Y, w, t) = D_0 - (4L^3/Ewt)\sqrt{(Y/t^2)^2 + (X/w^2)^2} \tag{8}$$

It is observed from Eqs. (7) and (8) that performance function G_1 is linear with respect to the random quantities, while G_2 is nonlinear. To compare the proposed RBDO method with the method by Wu et al. we consider the following four cases:

1) The allowable tip displacement D_0 is 2.2535 in., and the allowable system probability of failure P_f^{all} is 0.0027. (This corresponds to a system safety index of 2.782.) The solution to this design problem is compared to a design obtained by Wu et al.[32] in which the safety indices of the two failure modes β_1 and β_2 were constrained to be no less than $\beta_1^{\text{all}} = \beta_2^{\text{all}} = 3.0$, respectively. The formulation considered in the proposed approach and the one by Wu et al.[32] impose the same constraint on the system reliability because the maximum allowable values of the safety indices considered by Wu et al.[32] also correspond to a system safety index of 2.782. The difference is that the proposed approach lets the optimizer find the allowable minimum values of the safety indices of the failure modes instead of imposing arbitrary limits.

2) Case 2 is the same as case 1 but $P_f^{\text{all}} = 0.00135$, which corresponds to an equivalent system safety index of 3.0.

3) The allowable tip displacement D_0 is 2.50 in., and the allowable system probability of failure P_f^{all} is 0.0027. The solution to this design problem is also compared to a design obtained by Wu et al.[32] in which the safety indices of the two failure modes β_1 and β_2 were constrained to be no less than $\beta_1^{\text{all}} = \beta_2^{\text{all}} = 3.0$, respectively. These values correspond to a system safety index of 2.782.

4) Case 4 is the same as case 3 but $P_f^{\text{all}} = 0.00135$, which corresponds to an equivalent system safety index of 3.0.

First, we performed deterministic optimization with safety factor of 1.0 to obtain an initial design. Safety factor is the ratio of the mean capacity to the mean load. Then we designed the beam using the new system RBDO method. The area of the optimum design was compared to the area of the optimum design with the same system reliability obtained by Wu et al.[32] for cases 1 and 3. In this example, both failure modes are critical ($m_c = m = 2$). In each iteration, we performed PMA of the optimum design for the optimum values of the safety indices found from the semiprobabilistic optimization.

Figure 5 shows the iteration history for case 1. The optimum designs obtained by the proposed approach and by Wu et al.[32] are presented in the same figure. The results for the four cases are presented in Tables 2–5. For clarity, we have computed the safety indices of the two failure modes of the optimum using FORM. These are called "true" values of the safety indices in Tables 2–5.

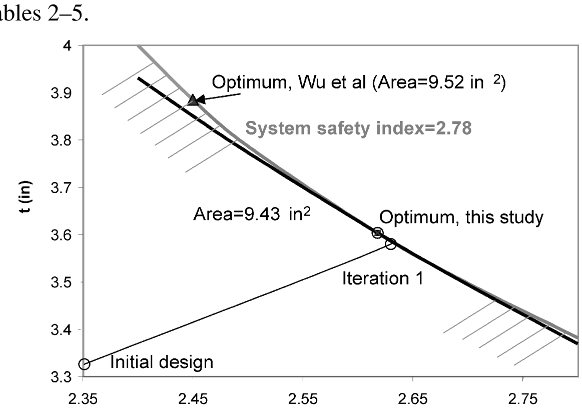

Fig. 5 Iteration history for example 1, case 1.

Table 2 Case 1: results for $D_0 = 2.2535$ in. and $P_f^{all} = 0.0027$

				1st iteration			2nd iteration				
Quantity	Deter. design	PMA G_1[a]	PMA G_2[a]	RBDO [Eq. (6)]	PMA G_1	PMA G_2	RBDO [Eq. (6)]	PMA G_1	PMA G_2	3rd iteration, RBDO [Eq. (6)]	Wu et al.
w, in.	2.3513	—	—	2.6328	—	—	2.6181	—	—	2.6208	2.4494
t, in.	3.3261	—	—	3.57984	—	—	3.604	—	—	3.6003	3.8886
X, lb	500	719	744	—	689.7	763	—	690.6	765.9	—	—
Y, lb	1,000	1,155	1,089	—	1139.5	1,112	—	1,138	1,107	—	—
E, psi	$2.9E7$	$2.90E7$	$2.68E7$	—	$2.90E7$	$2.64E7$	—	$2.90E7$	$2.64E7$	—	—
R, psi	40,000	37,315	40,000	—	36,862	40,000	—	36,861	40,000	—	—
Area, in.^{2}	7.821	—	—	9.425	—	—	9.436	—	—	9.436	9.5247
β_1 (Projected)	—	—	—	2.830	—	—	2.831	—	—	2.830	—
(True)	0.13878	—	—	2.808	—	—	2.831	—	—	2.830	3.0071
β_2 (Projected)	—	—	—	3.373	—	—	3.368	—	—	3.373	—
(True)	0.00583	—	—	3.376	—	—	3.367	—	—	3.373	3.0097
P_f	0.9425	—	—	0.002862	—	—	0.002701	—	—	0.0027	0.0027

[a]A safety index of 3.0 was used to perform the PMA analysis of the deterministic design.

Table 3 Case 2: results for $D_0 = 2.2535$ in. and $P_f^{all} = 0.00135$

				1st iteration			2nd iteration			
Quantity	Deter. design	PMA G_1[a]	PMA G_2[a]	RBDO [Eq. (6)]	PMA G_1	PMA G_2	RBDO [Eq. (6)]	PMA G_1	PMA G_2	3rd iteration, RBDO [Eq. (6)]
w, in.	2.35132	—	—	2.640	—	—	2.627	—	—	2.628
t, in.	3.32610	—	—	3.616	—	—	3.639	—	—	3.639
X, lb	500	734	760	—	703	783	—	703	785	—
Y, lb	1,000	1,165	1,094	—	1,148	1,114	—	1,147	1,110	—
E, psi	$2.9E7$	$2.9E7$	$2.66E7$	—	$2.9E7$	$2.62E7$	—	$2.9E7$	$2.62E7$	—
R, psi	40,000	37,130	40,000	—	36,592	40,000	—	36,589	40,000	—
Area, in.^{2}	7.821	—	—	9.547	—	—	9.561	—	—	9.561
β_1 (Projected)	—	—	—	3.037	—	—	3.037	—	—	3.037
(True)	0.13878	3.2053	—	3.010	3.037	—	3.037	3.037	—	3.037
β_2 (Projected)	—	—	—	3.610	—	—	3.604	—	—	3.605
(True)	0.00583	—	3.2053	3.607	—	3.610	3.604	—	3.604	3.605
P_f	0.94249	—	—	0.00146	—	—	0.00135	—	—	0.00135

[a]A safety index of 3.2053 was used to perform the PMA analysis of the deterministic design.

Table 4 Case 3: results for $D_0 = 2.5$ in. and $P_f^{all} = 0.0027$

				1st iteration			2nd iteration				
Quantity	Deter. design	PMA G_1[a]	PMA G_2[a]	RBDO [Eq. (6)]	PMA G_1	PMA G_2	RBDO [Eq. (6)]	PMA G_1	PMA G_2	3rd iteration, RBDO [Eq. (6)]	Wu et al.
w, in.	2.0470	—	—	2.4708	—	—	2.4768	—	—	2.4762	2.4573
t, in.	3.7460	—	—	3.7959	—	—	3.7913	—	—	3.7921	3.8743
X, lb	500	719	744	—	696	822	—	696	824	—	—
Y, lb	1,000	1,155	1,089	—	1,128	1,076	—	1,128	1,077	—	—
E, psi	$2.9E7$	$2.9E7$	$2.68E7$	—	$2.90E7$	$2.62E7$	—	$2.9E7$	$2.62E7$	—	—
R, psi	40,000	37,312	40,000	—	36,967	40,000	—	36,964	40,000	—	—
Area, in.^{2}	7.6681	—	—	9.379	—	—	9.3902	—	—	9.3902	9.5203
β_1 (Projected)	—	—	—	2.7900	—	—	2.7894	—	—	2.7895	—
(True)	0.00012	3.00	—	2.7717	2.7900	—	2.7894	2.7894	—	2.7895	3.0
β_2 (Projected)	—	—	—	3.8264	—	—	3.84708	—	—	3.8449	—
(True)	0.00015	—	3.00	3.8077	—	3.8264	3.84706	—	3.84708	3.8449	3.9478
P_f	1.0	—	—	0.002705	—	—	0.0027	—	—	0.0027	0.00139

[a]A safety index of 3.00 was used to perform the PMA analysis of the deterministic design.

Discussion of the Results

The reliability of the optimum RBDO design in Table 2 was also estimated using Monte Carlo simulation. An importance sampling approach was used to generate 1000 sample values of the random variables. The sampling distributions were selected in a way that a large number of failures occurred in the simulation. The safety indices of the two failure modes were found equal to 2.84 and 3.18, respectively, and the system failure probability was found equal to 0.0028. These results agree well with those in Table 2.

From Tables 2–5 we observe that the deterministic optimization with a safety factor of one yielded very unsafe designs. Table 2 shows that the proposed system RBDO approach yielded a design with area (and consequently weight) 0.93% lower than that obtained by Wu et al.[32] for the same requirements on system reliability. Although the two competing final designs in Table 4 do not have same system reliability, both designs were designed for the same requirement for system reliability (0.0027). However, because Wu et al.'s design approach does not allow the designer to control directly the system reliability, it resulted in overdesign;

Table 5 Case 4: results for $D_0 = 2.50$ in. and $P_f^{\text{all}} = 0.00135$

Quantity	Deter. design	PMA G_1[a]	PMA G_2[a]	1st iteration RBDO [Eq. (6)]	1st iteration PMA G_1	1st iteration PMA G_2	2nd iteration RBDO [Eq. (6)]	2nd iteration PMA G_1	2nd iteration PMA G_2	3rd iteration, RBDO [Eq. (6)]
w, in.	2.0470	—	—	2.4856	—	—	2.4909	—	—	2.4904
t, in.	3.7460	—	—	3.8261	—	—	3.8237	—	—	3.8246
X, lb	500	734	760	—	710	842	—	709	843	—
Y, lb	1,000	1,165	1,093	—	1,137	1,078	—	1,137	1,079	—
E, psi	$29E6$	$2.9E7$	$2.66E7$	—	$2.9E7$	$2.6E7$	—	$2.9E7$	$2.6E7$	—
R, psi	40,000	37,130	40,000	—	36,687	40,000	—	36,681	40,000	—
Area, in.^{2}	7.6681	—	—	9.510	—	—	9.5245	—	—	9.5245
β_1 (Projected)	—	—	—	3.0053	3.0053	—	3.0049	3.0049	—	3.0049
(True)	0.00012	3.2053	—	2.9821	—	—	3.0049	—	—	3.0049
β_2 (Projected)	—	—	—	4.0702	—	4.0702	4.0922	—	4.0922	4.0899
(True)	0.00015	—	3.2053	4.0505	—	—	4.0922	—	—	4.0899
P_f	1.0	—	—	0.00147	—	—	0.00135	—	—	0.00135

[a]A safety index of 3.2053 was used to perform the PMA analysis of the deterministic design.

the final design has a safety index of 2.991 instead of the required one of 2.782. Table 4 shows that the new approach achieves a weight savings of 1.37% for the same requirement on the system reliability.

Tables 3 and 5 also show that the proposed RBDO approach enabled us to reduce the system probability of failure by 50% of the designs obtained by Wu et al.,[32] by increasing their weight by only 0.38 and 0.044%, respectively. (Compare with Wu et al.[32] results in Tables 2 and 4.) It is difficult to achieve these savings using a design approach that constrains the reliabilities of individual failure modes instead of the system reliability because this approach does not allow the designer to control directly the system reliability.

Finally, the results in Fig. 5 and Tables 2–5 demonstrate the fast convergence of the method because we needed only three PMA reliability analyses and three approximate deterministic optimizations to find the optimum design. The convergence can be slower for systems with more failure modes and random variables.

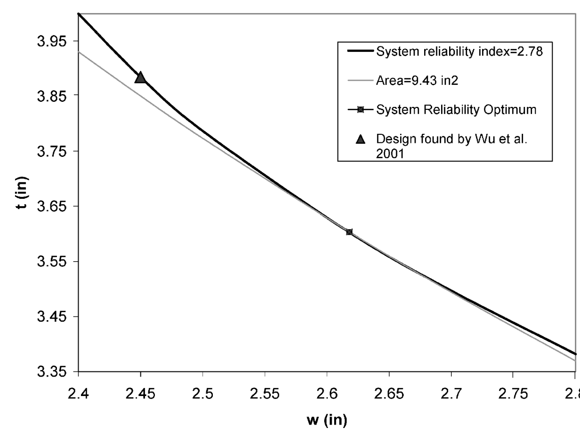

Fig. 6 Optimality condition for beam design: any deviation from system reliability optimum results in a heavier or less reliable design.

Optimality Condition of System RBDO and Validation of the Results

Consider the system RBDO problem formulation in this example:

$$\begin{aligned} &\text{Find} \quad w \quad \text{and} \quad t \\ &\text{to min} \quad A = w \cdot t \\ &\text{so that} \quad \beta_{\text{sys}} \geq \beta_{\text{sys}}^{\text{all}} \end{aligned} \tag{9}$$

The Lagrangian of the preceding constrained optimization problem is

$$L = A + \lambda\left(\beta_{\text{sys}}^{\text{all}} - \beta_{\text{sys}}\right) \tag{10}$$

At the optimum, the constraint on the system safety index is usually active, that is, the system reliability index assumes its minimum allowable value to minimize the weight. Then, the optimality conditions become

$$\begin{aligned} \frac{\partial L}{\partial w} &= 0 \Rightarrow \frac{\partial A}{\partial w} = \lambda \frac{\partial \beta_{\text{sys}}}{\partial w} \\ \frac{\partial L}{\partial t} &= 0 \Rightarrow \frac{\partial A}{\partial t} = \lambda \frac{\partial \beta_{\text{sys}}}{\partial t} \end{aligned} \tag{11}$$

From Eq. (11) we obtain the following optimality criterion for the system RBDO problem:

$$\frac{\partial A / \partial w}{\partial A / \partial t} = \frac{\partial \beta_{\text{sys}} / \partial w}{\partial \beta_{\text{sys}} / \partial t} \tag{12}$$

Equation (12) says that, at the optimum, the safety indices (and consequently the reliabilities) of the failure modes do not have to be equal. Rather, at the optimum, the isocost curves (loci of all designs with constant area A) and isoreliability curves (loci of all designs with same system reliability) have the same slope, that is, they are tangent to each other. We will check whether the optimum in Table 2 satisfies the preceding optimality condition.

Figure 6 compares the optimum designs obtained by the proposed system RBDO approach and the approach by Wu et al.[32] for case 1. It is observed that the system reliability optimum has the smallest area among all of the designs with same system reliability ($P_f = 0.0027$, system reliability = 1–0.0027, system safety index = 2.782). Isoreliability curves (curves representing designs with system reliability 1–0.0027) and isocost curves (designs with area = 9.43 in.^{2}) are tangent at the point representing the optimum reliability-based design. Any deviation from this optimum will result in a design with larger area or an infeasible design (violation of the minimum system reliability constraint). The design by Wu et al.[32] lies on the same isoreliability curve as the RBDO optimum, but it has higher area. This shows that the proposed approach saves area by apportioning reliability in an optimal way among the failure modes of a system.

Example 2: Design of a Cable-Stayed Box Girder

This example is similar to one considered by Nikolaidis and Burdisso.[13] The aim is to apply the method to further practical design examples with multiple constraints and to determine whether the proposed RBDO method would be able to provide an optimum design with strategic distribution of the risk of failure over the failure modes of the system. Also, we aim to examine the convergence of the method and to demonstrate the computational savings of the proposed method.

The cable stayed box girder shown in Fig. 7 has five failure modes G_1–G_5 [Eqs. (13–17)]. The objective is to minimize the cost of this structure [Eq. (18)] by controlling the thicknesses of the flange t_f and the web t_w and the cable area A_c. The maximum allowable system probability of failure is 10^{-4}.

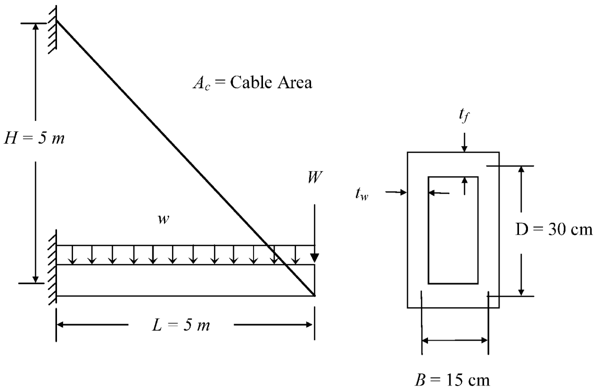

Fig. 7 Cable-stayed box girder.

The first constraint is on the maximum direct and induced compressive stress in the beam and is given by

$$G_1 = 1 - |f_a/S_c| - |f_b/S_{\text{bc}}| \tag{13}$$

where f_a is the direct compressive stress and f_b is the bending stress. The second and third constraints are on the maximum shear stresses in the two ends of the beam and are given by

$$G_2 = 1 - |f_s/S_s| \tag{14}$$

$$G_3 = 1 - |f_s'/S_s| \tag{15}$$

where f_s and f_s' are the shear stresses at the free and the fixed ends of the beam, respectively. The fourth constraint is on the maximum tensile stress in the supporting cable and is given by

$$G_4 = 1 - |f_t/S_{\text{tc}}| \tag{16}$$

where f_t is the tension stress at the cable. The fifth constraint is on the maximum allowable vertical deflection of the right end of the beam and is given by

$$G_5 = 1 - \delta_u/\delta_{\max} \tag{17}$$

where δ_u and $\delta_{\max}$ are the vertical deflection and the maximum allowable vertical deflection of beam, respectively.

The cost of the beam is assumed proportional to the sum of the volume of the material of the beam plus the volume of the material of the cable scaled by five. Thus, we want to minimize the following objective function:

$$F = 2(Bt_f + Dt_w)L + 5A_c(H^2 + L^2)^{\frac{1}{2}} \tag{18}$$

The applied loads and the resisting strengths are random quantities that are assumed normally distributed and statistically independent with statistics listed in Table 6.

In this example we found an initial design by performing deterministic optimization for a factor of safety of one. The optimum design is shown in the second column of Table 7. Then, we performed reliability analysis (using a FORM) to determine the critical failure modes of the deterministic design. The values of the safety indices of these modes are shown in the second column of Table 7. Failure modes G_3 to G_5 contribute less than 1% to the system probability of failure. To facilitate convergence, we removed the minimum allowable values of the safety indices of these noncritical modes β_3^{all} to β_5^{all} from the set of design variables for the first approximate deterministic optimization along with the corresponding constraints. However, after this iteration we calculated the system failure probability considering the contribution of all five failure modes and compared it to the maximum allowable system failure probability.

In the first iteration, we assigned equal values to the safety indices for the two critical modes so that the system probability of failure is equal to the maximum acceptable value $P_f^{\text{all}} = 10^{-4}$ (for $\beta_1^{\text{all}} = \beta_2^{\text{all}} = 3.891$). The MPPs for the first two critical failure modes obtained from PMA analysis are shown in the third and fourth columns of Table 7. We performed the first approximate deterministic optimization and obtained the optimum values of safety indices $\beta_1^{\text{all}} = 3.72$, $\beta_2^{\text{all}} = 4.80$. Next, we performed FORM reliability analysis of the obtained deterministic semiprobabilistic optimization (DSPO) design, and we verified that modes G_3 to G_5 have a considerably lower effect on the safety of the design than modes G_1 and G_2.

For the second iteration (Table 7), we performed PMA to determine the MPPs of the two failure modes G_1 and G_2 for the projected values of the safety indices of these modes from the first DSPO iteration. Then, we performed the second approximate deterministic optimization to determine the optimum minimum allowable values of the safety indices of the first two failure modes and the corresponding values of the design variables. By comparing the optimum designs from the first two iterations in Table 7, we observe that the design did not change appreciably. Moreover, the projected values of the safety indices were almost equal to the ones obtained from FORM. Finally, we found that the system reliability is approximately equal to 10^{-4}, which is acceptable. However, to check for convergence we performed a third iteration, which yielded practically identical results as the second.

Finally, to check for the convergence of the algorithm we repeated the RBDO starting from the deterministic optimum design with safety factor 1.5, and we obtained the same optimum design as the one in Table 7.

Table 6 Mean values and the coefficients of variation for the random quantities

Quantity	Mean value	COV, %
Concentrated load W, N	400,125	20
Distributed load w, N/m	78,546	20
Maximum allowable bending stress S_{bc}, MPa	154.5	20
Maximum allowable compressive stress S_c, MPa	122.59	20
Maximum allowable shear stress S_s, MPa	92.68	20
Maximum allowable cable stress S_{tc}, MPa	1,176.84	20

Validation of the Optimum

To check whether the found optimum design is correct, we 1) computed the system failure probability of the optimum design using numerical integration and compared it with the estimate of the failure probability of the same design in Table 7 and 2) compared the found optimum design against perturbed designs around the optimum. The first check tested the accuracy of the FORM algorithm employed by the proposed approach to estimate the system failure probability, whereas the second check tested whether the found optimum is a true local optimum.

The failure probabilities of the five failure modes can be calculated by integrating the joint probability density function of the random variables over the failure domain for each mode. The failure probability of the first failure mode can be computed using the total probability theorem as follows:

$$P(F_1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(F_1/w, W, S_c) \times f_w(w) f_W(W) f_{S_c}(S_c)\,\mathrm{d}w\,\mathrm{d}W\,\mathrm{d}S_c \tag{19}$$

where $P(F_1/w, W, S_c)$ is the conditional failure probability under bending given the values of the w, W, and S_c. From Eq. (13) we find that failure under the first mode occurs when

$$1 \le \frac{|f_a|}{S_c} + \frac{|f_b|}{S_{\text{bc}}} \Leftrightarrow S_{\text{bc}} \le \frac{|f_b|}{1 - |f_a|/S_c} \tag{20}$$

Therefore, the conditional probability of failure under the first mode is

$$P(F_1/w, W, S_c) = F_{S_{\text{bc}}}\left(\frac{|f_b|}{1 - |f_a|/S_c}\right) \tag{21}$$

where $F_{S_{\text{bc}}}[|f_b|/(1 - |f_a|/S_c)]$ is the cumulative probability distribution function of the maximum allowable compressive bending

Table 7 RBDO for the cable-stayed box girder (iterations 1 and 2)

		1st iteration			2nd iteration		
Quantity	Det	PMA G_1	PMA G_2	RBDO	PMA G_1	PMA G_2	RBDO
t_w, m	0.005046	——	——	0.08407	——	——	0.08428
t_f, m	0.088742	——	——	0.27709	——	——	0.2781
A_c, m^2	0.002844	——	——	0.013995	——	——	0.01406
W, KN	400	423	406.5	——	427	401.8	——
w, KN/m	78.5	87.3	89.77	——	87.75	81.16	——
S_{bc}, MPA	154.5	35.86	154.5	——	41.44	154.5	——
S_c, MPA	122.6	121.8	122.6	——	121.77	122.6	——
S_s, MPA	92.7	92.68	21.8	——	92.68	3.74	——
S_{tc}, MPA	1176.8	1176.8	1176.8	——	1176.8	1176.8	——
F, m^3	0.24881	——	——	1.16265	——	——	1.16688
β_1 (projected)	——	3.891[a]	——	3.7211	3.7211[b]	——	3.7250
(FORM)	0.0002	——	——	3.7202	——	——	3.7250
β_2 (projected)	——	——	3.891[a]	4.8016	——	4.8013[b]	4.7540
(FORM)	0.0003	——	——	4.7529	——	——	4.7539
β_3 (FORM)	2.79	——	——	4.9139	——	——	4.9142
β_4 (FORM)	3.87	——	——	4.782	——	——	4.782
β_5 (FORM)	100.2	——	——	525.6	——	——	526
$P_{f(\text{sys})}$	1.0	——	——	0.000102	——	——	9.9994E-05

[a]Assuming equal safety indices for the first two failure modes [Eq. (6a)].
[b]The projected value of the safety index obtained from iteration 1.

Table 8 Comparison of probabilities of failure of the modes of the optimum design and the system failure probability obtained from FORM in the proposed approach and from numerical integration

Failure probability	RBDO	Numerical integration
Mode 1	9.7661×10^{-5}	9.7134×10^{-5}
Mode 2	9.9674×10^{-7}	9.9525×10^{-7}
Mode 3	4.2238×10^{-7}	4.4517×10^{-7}
Mode 4	8.6503×10^{-7}	8.6142×10^{-7}
Mode 5	0	0
System	9.995×10^{-5}	9.944×10^{-5}

stress. This is a normal distribution with mean and coefficient of variation specified in Table 6. Therefore, the probability of failure under bending is

$$P(F_1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_{S_{\text{bc}}}\left(\frac{|f_b|}{1-(|f_a|/S_c)}\right) \times f_w(w) f_W(W) f_{S_c}(S_c)\,\mathrm{d}w\,\mathrm{d}W\,\mathrm{d}S_c \tag{22}$$

The failure probabilities of the remaining four failure modes can be found in a similar way.

Although we can compute the failure probabilities of the five failure modes, it is impractical to compute the system failure probability using numerical integration. Indeed, the latter computation requires integration of the joint probability density function of all six random variables over the failure region for the system. The sixfold nested integral is too expensive to compute numerically. Therefore, we approximated the system failure probability with the first-order Ditlevsen upper bound. (That is, the system failure probability is equal to the sum of the failure probabilities of the five failure modes.)

Table 8 compares the values of the probabilities of failure of the optimum design obtained by the proposed RBDO method and by numerical integration. It is observed from the preceding table that the results that the reliability assessment of the optimum design obtained from the proposed RBDO method and from FORM agree well. The system failure probability from the RBDO approach is greater than the same probability obtained by numerical integration by 0.5%.

To check whether the obtained optimum is a true local optimum, we can compare its cost to those of perturbed designs around the optimum that have the same system failure probability. A practical alternative approach is to compare the system failure probability of

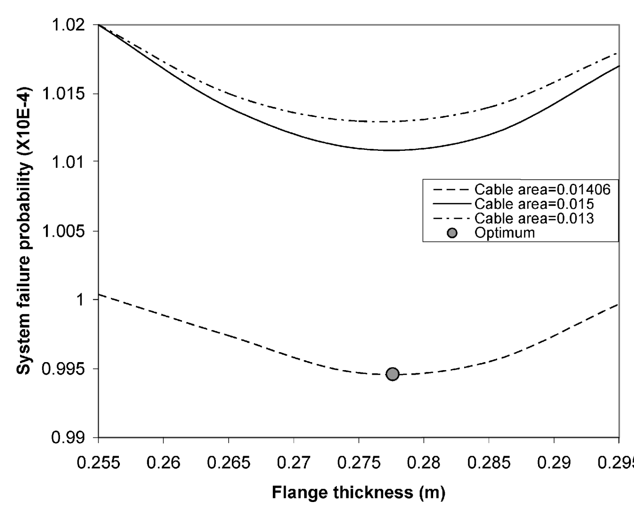

Fig. 8 Comparison of sysem failure probability of found optimum to those of 15 designs found by perturbing the design variables around the optimum. All designs in the figure have same cost as that of the found optimum: Isocost lines, objective function = 1.16688 m^3.

the found optimum to those of the perturbed designs with the same cost. These designs were obtained by perturbing the design variables around the optimum values so that the cost function remained the same as that of the obtained optimum. Specifically, we considered all 15 designs corresponding to all of the combinations of values for the cable area A_c, 0.013, 0.01406, and 0.015 m^2, and values of the flange thickness, 0.255, 0.265, 0.275, 0.285, and 0.295 m. The web thickness was computed by solving Eq. (18) for t_w when $F = 1.16688$ m^3. Because the found optimum has same objective function, if it is a true local optimum then it ought to have lower system failure probability than the preceding 15 designs. Figure 8 compares the system failure probability of the optimum to those of the 15 perturbed designs. The obtained optimum design has the lowest system failure probability among all designs with the same cost in the figure. This suggests that the found optimum is true local one.

Discussion of the Results

From Table 7, we observe a large difference in the costs and in the reliabilities of the deterministic and the reliability-based optima. In addition, we see that the change of the design variables is not uniform, as the five different constraints impose different requirements for the design variables to reach their target safety levels. Also, Table 7 shows that the method converged in just two iterations, involving two reliability analyses and an approximate deterministic optimization, which suggests that the proposed approach is efficient.

Summary

The computational cost of most of the probabilistic optimization methods has limited their applicability and their acceptance by the industry as a complete substitute for currently used deterministic optimization methods. Sequential optimization with reliability-based safety factors techniques (SORFS) can reduce dramatically the required calculations. However, existing SORFS RBDO techniques do not allow the designer to control directly the overall system reliability. In particular, no attempt has been made to find an optimum distribution for the risks of failure among the failure modes. Accordingly, we have modified Du and Chen's[33] technique to constrain only the system reliability of a system in RBDO and let the optimizer distribute the reliability of the system over its components in an optimal way. The key idea is to optimize simultaneously the system design variables and the allowable failure probabilities of the system modes. We have applied this proposed approach on two examples. One example considered a cantilever beam of a rectangular cross section under lateral and vertical loads with two failure modes, and the other example considered a cable-stayed box girder with five failure modes. The results showed that the new system RBDO approach can provide better optimum designs than the ones in which the distribution of the risks of failure is not taken into account. Also, the results showed that the new RBDO method for systems requires only a moderate increase in computational expense over that for the deterministic optimum design.

As with many optimization methods, we do not expect this new RBDO method to perform as well for all highly nonlinear problems. In such cases, we recommend performing sensitivity analysis for the design variables and use adequate move limits for those design variables that affect most the cost and the reliability. Also, we recommend using the deterministic optimization first to locate roughly the desired level of safety and then using our method to increase the cost (and/or weight) savings or increase the reliability of the design.

Future research can focus on extending the proposed method to single-loop optimization methods.

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